

Matrices:

A system of 3 linear equation in 3 variables represents 3 planes in 3d space

$$ax + by + cz = d$$

$$ex + fy + gz = h$$

$$ix + jy + kz = l$$

Then in matrices form

$$\begin{bmatrix} a & b & c & | & d \\ e & f & g & | & h \\ i & j & k & | & l \end{bmatrix}$$

i) When

- $d \neq 0$ or $h \neq 0$ or $l \neq 0 \rightarrow$ Non-Homogenous Equations

After changing it to echelon form following is true

- $i = j = k = 0 = l$
- Then there are infinitely many solutions
- 3rd plane is just the rotated version of 2nd plane about the intersection line of 2nd and 1st planes
- x and y are linearly dependent on z
- One of three variables is a free variable, you can assume it to be 't'
- 3rd equation is some linear combination of 1st and 2nd meaning that multiplying 1st or 2nd with some constant and adding it to 2nd equation makes it the 3rd equation
- The intersection of 3 planes is a straight line
- *Rank of A = Rank of A_b ≠ number of unknowns*

ii) When

- $d \neq 0$ or $h \neq 0$ or $l \neq 0 \rightarrow$ Non-Homogenous Equations

After changing it to echelon form following is true

- $k = 1$
- Then there is only one unique Solution
- Three planes intersect at a single point in space, so there is only one value of x, y and z at which all equations are true.
- *Rank of A = Rank of A_b = number of unknowns*

iii) When

- $d = h = l = 0 \rightarrow$ Homogenous Equations, Planes are passing through the origin

After changing it to echelon form following is true

- *Rank of A = Rank of A_b = number of unknowns*
- Then $x = y = z = 0$
- Origin is the point of intersection of three planes

iv) When

- $d = h = l = 0 \rightarrow$ Homogenous Equations, Planes are passing through the origin

After changing it to echelon form following is true

- *Rank of A = Rank of A_b ≠ number of unknowns*
- One of the rows has all zeros
- There are infinitely many solutions
- The solution is thus a straight line passing through the origin
- x and y are linearly dependent on z
- x, y or z is a free variable

- The 3rd equation is some linear combination of first two equations i.e. multiplying equation one or two with some constant and adding it to eq. one or two forms the third equation
- Solution is called non-trivial solution
- 3rd plane is rotated version of other two planes about the line of intersection of 1st and 2nd plane
- The equation can be represented as the dot product of rows with the unknown vector $\langle x, y, z \rangle$:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} e \\ f \\ g \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} i \\ j \\ k \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

This means the vectors $\langle a, b, c \rangle$, $\langle e, f, g \rangle$, $\langle i, j, k \rangle$ are perpendicular to the vector $\langle x, y, z \rangle$ hence the three vectors form a plane which is perpendicular to $\langle x, y, z \rangle$, this plane is called ROW SPACE.

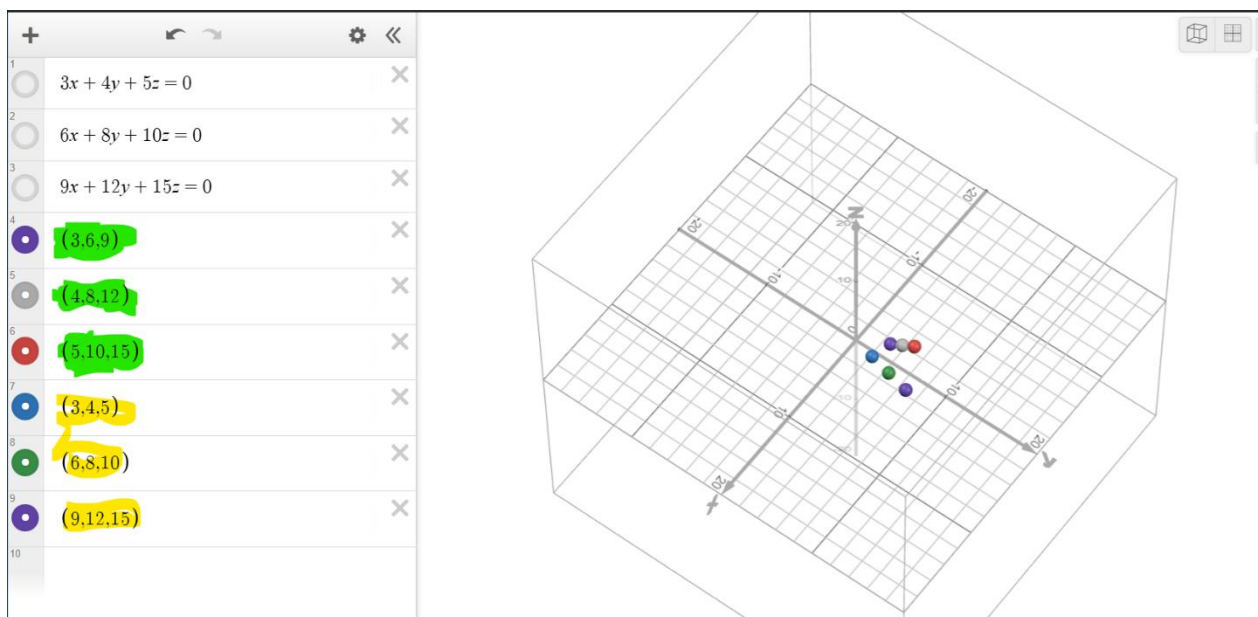
- Another way is as follows:

$$x \begin{pmatrix} a \\ e \\ i \end{pmatrix} + y \begin{pmatrix} b \\ f \\ j \end{pmatrix} + z \begin{pmatrix} c \\ g \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- $\langle a, e, i \rangle$, $\langle b, f, j \rangle$ and $\langle c, g, k \rangle$ are vectors and they are scaled by x, y and z times respectively. The sum results in null vector hence after scaling the vectors form a closed triangle which has one corner at origin because there is no constant added in equations, the equations are not shifted in any axis. The plane in which this triangle is is called the column space
- Both row and column space are always same dimensional and their dimension is equal to the rank of matrix. i.e.

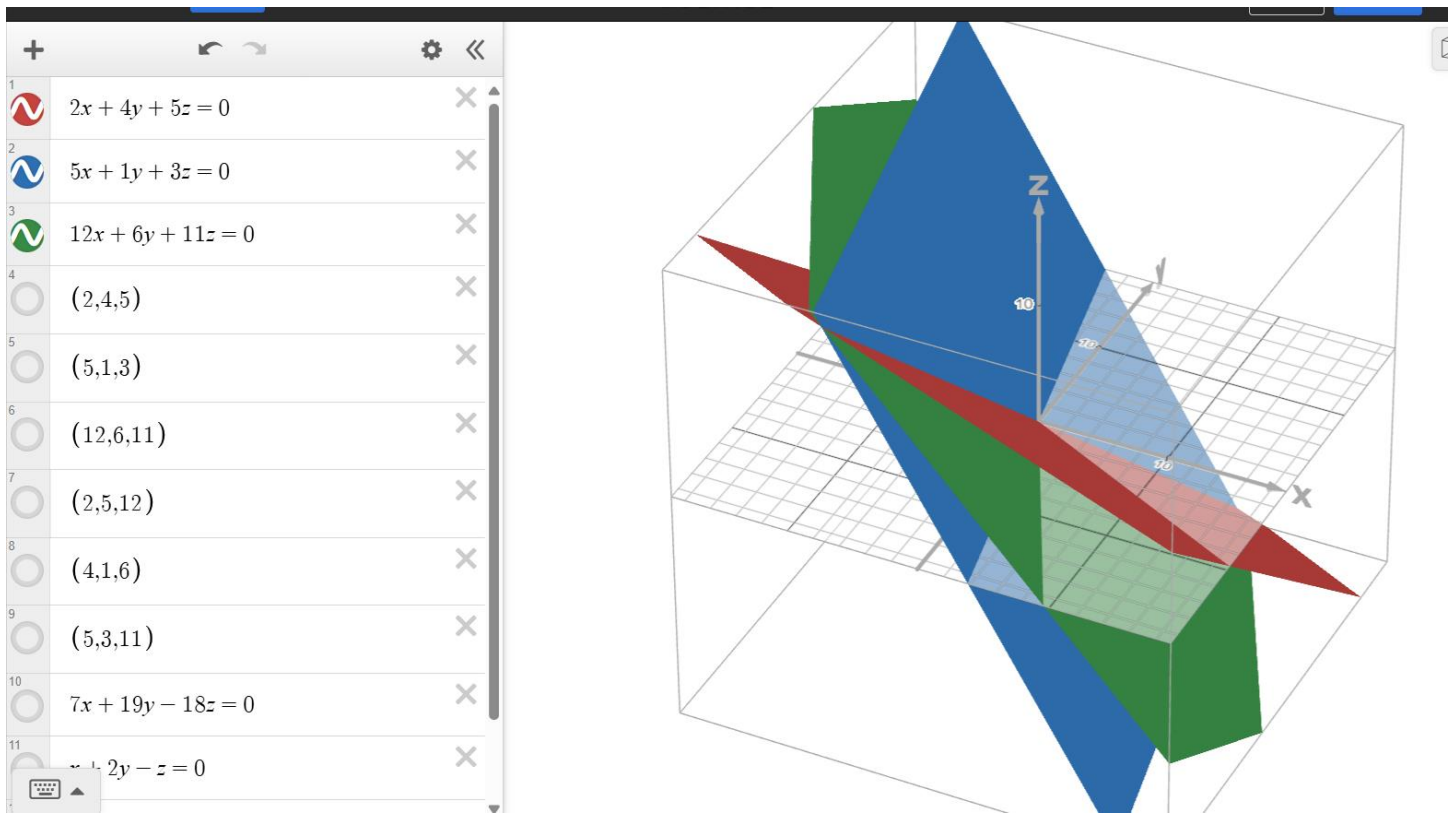
$$\text{rank of matrix} = \text{dimension of column space} = \text{dimension of row space}$$

- Rank = 1, means all three equations represent same plane. Hence the column vectors and row vectors are in line and the row space and column space is a line(1D). Such as follows:

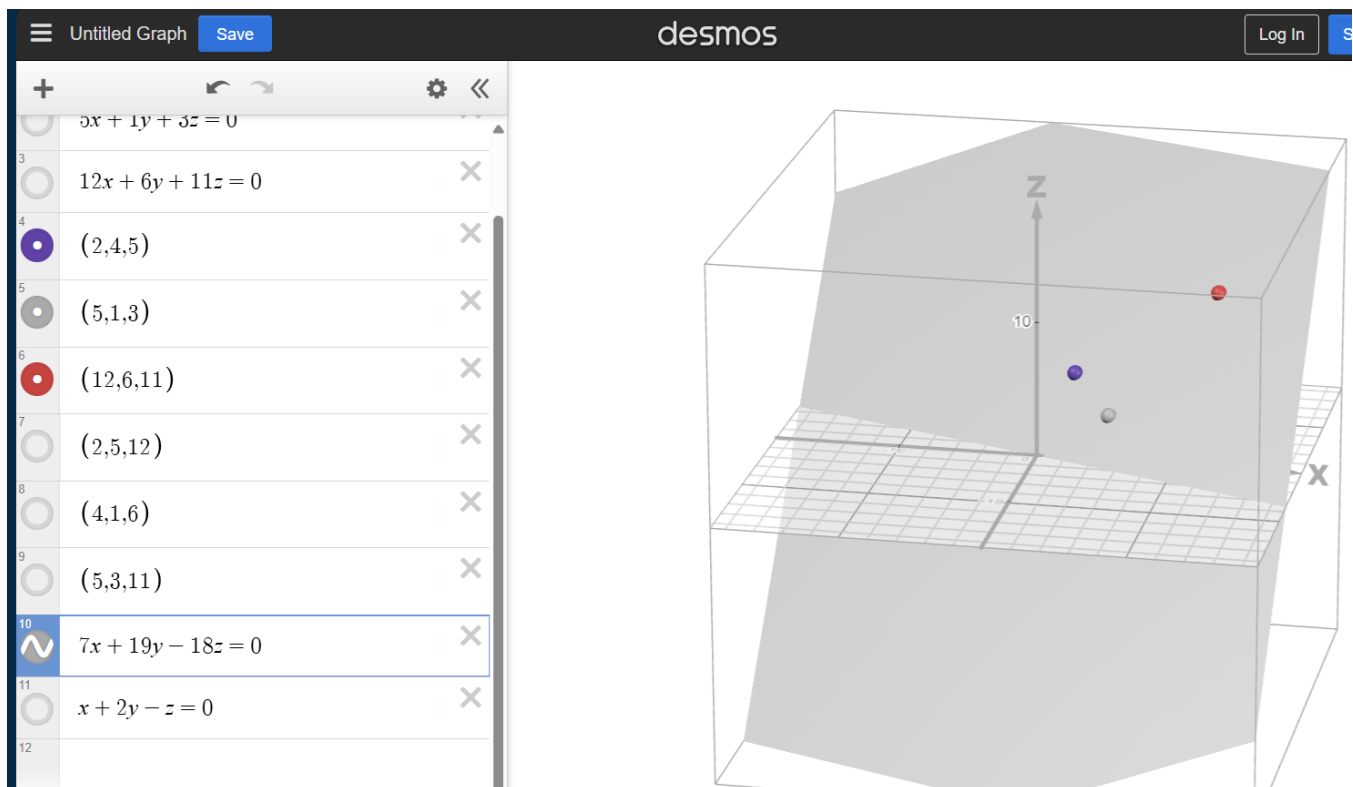


These three vectors can scale up and add in each other but they will not move away in any other direction except the 1-dimensional line. This line is column space now. You can see in figure that column space and row space are different lines but same dimension. Green highlighted is column vectors and yellow Highlighted is Row space.

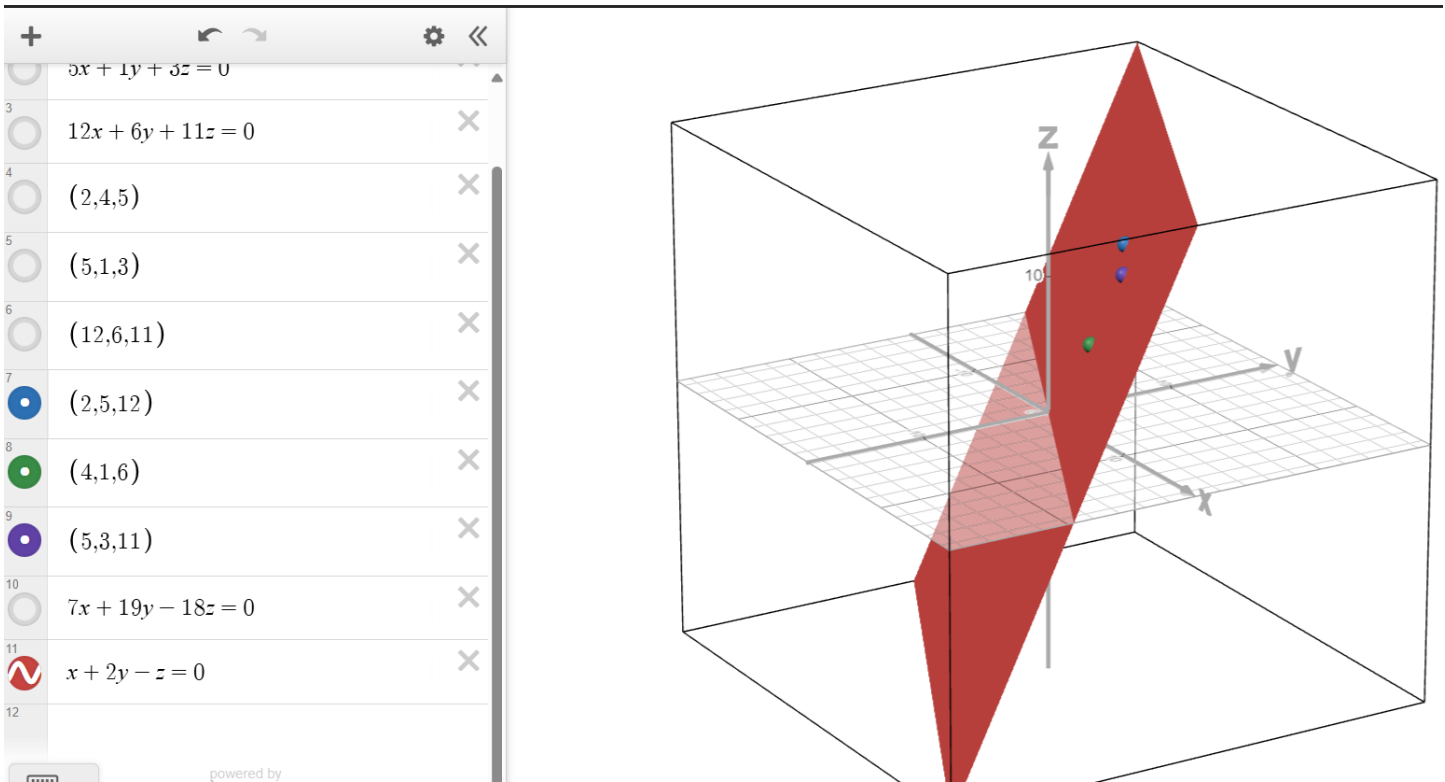
- In case of rank = 3, the only solution to column vectors equation is that x, y, z are all zeros. This is called trivial solution and column space and row space is 3-dimensional because three independent vectors can direct in any direction and hence they span 3-dimensions
- In case of Rank = 2 such as:



The Three planes intersect to form a line which is the line of solution and called null space and the line is passing through the origin.



The three row vectors and origin are in a single plane called row space. You can scale any of these vectors and add in any other to get some vector in this plane. You can not get out of this plane by adding or scaling then adding the three vectors
Same is the case with the Column vectors such as shown



- Null space is the span of $\langle x, y, z \rangle$ vector. When $x = y = z = 0$ the null space is 0 dimensions. And when rank is 2 null space is 1-Dimension since the solution is any where in straight line and x, y are related to z .

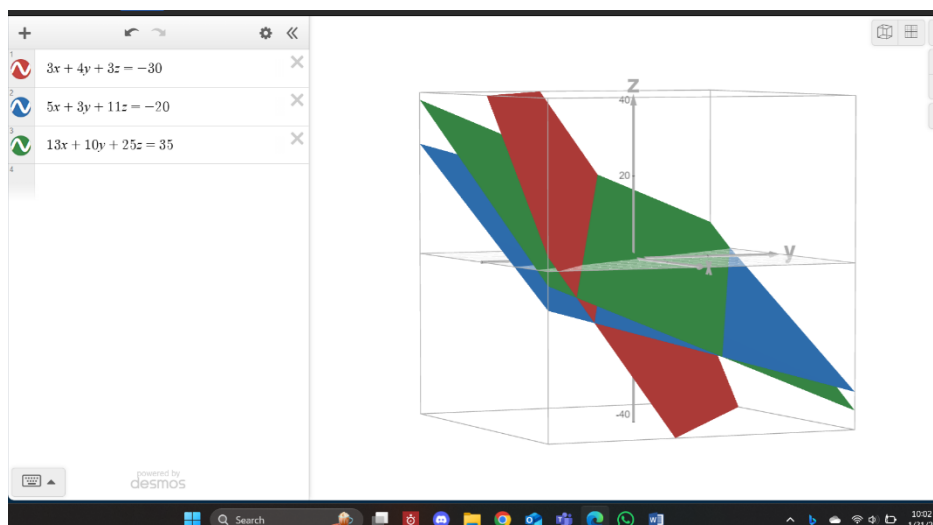
v)

When

- $(d = h = l = 0)$ OR $(d \neq 0 \text{ or } h \neq 0 \text{ or } l \neq 0)$

After changing it to echelon form following is true

- $(d \neq 0 \text{ or } h \neq 0 \text{ or } l \neq 0)$ And *(one of three rows is all zero)*
- Rank of A $\neq$ Rank of A_b*
- The system is inconsistent*
- The three planes intersect and may form three or two different intersecting lines.



Methods to make matrix's echelon form

- I) Avoid running into fractions
- II) Swap rows to make $a_{11} = 1$ or -1
- III) Subtract from first row any other row such that difference is 1 or -1
- IV) Divide the first row with the a_{11} (Least Practiced way, makes a fraction)